ON THE STABILITY OF MOTION

(OB USTOICHIVOSTI DVIZHENIIA)

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V.M. MATROSOV
(Kazan')

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In this paper sufficient indications of asymptotic stability and instability are obtained, generalizing the known criteria of Liapunov [1] by replacing the condition of sign-definiteness of the derivative of the Liapunov function by a less rigorous condition of its uniformity of sign (with some requirements for the set where the derivative becomes zero).
comes zero).

Generalizations of this type were obtained in the case of steady motion (in the sense of [1]) by Barbashin and Krasovskii [2] and also by Tuzov [3], and for the case of periodic motions by Krasovskii [4]. Here the general case of nonsteady motion is considered. It is easy to convince oneself by example*, that the generalizations of the mentioned authors, in the forms [2,3,4] do not extend to that case. In the obtained criteria two Liapunov functions are used. As is known, the first theorem on instability with two functions was proposed by Chetaev [5]. For the case of nonuniform asymptotic stability, the requirement of an infinitely small higher limit is also removed, which leads to the modification of

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* Thus

$$\frac{dx}{dt} = -p(t)x \qquad \left(p(t) \geqslant 0, \int_{t_0}^{\infty} p\,dt < \infty\right)$$

The general solution

$$x = x_0 \exp\left[-\int_{t_0}^{t} p\,dt\right]$$

shows that the solution $x = 0$ is nonasymptotically stable, although for $V = 1/2\ x^2$, the derivative $\dot{V} = -px^2 \leqslant 0$ and the semi-trajectories, with the exception of $x = 0$, are not contained entirely in the set, where $\dot{V} = 0$.

the corresponding theorems of Krasovskii [6], Zubov [7], Reisig [8].

The application to nonstationary gyroscopic systems with dissipation is discussed.

1. Let the equations of the nonautonomous motion be given by

$$\frac{dx_i}{dt} = X_i(x_1, \ldots, x_n, t) \qquad (i = 1, \ldots, n) \tag{1.1}$$

in which the functions X_i in the domain Γ

$$x_1^2 + \ldots + x_n^2 < H^2, \qquad t \geqslant 0 \qquad (H = \text{const} > 0)$$

are defined, continuous and bounded as well as their partial derivatives $\partial X_i/\partial x_j$, $\partial X_i/\partial t$, as

$$|X_i(x_1, \ldots, x_n, t)| < X \qquad (X = \text{const} > 0)$$

In such a case, to each set of numbers $(x_{10}, \ldots, x_{n0}, t_0) \in \Gamma$, there corresponds a single system of functions which are continuously differentiable with respect to t

$$x_i\,(t, x_{10}, \ldots, x_{n0}, t_0) \qquad (i = 1, \ldots, n)$$

and satisfying in Γ to the system (1.1) and to the initial conditions

$$x_i\,(t_0, x_{10}, \ldots x_{n0}, , t_0) = x_{i0} \qquad (i = 1, \ldots, n)$$

Let also

$$X_i\,(0, \ldots, 0, t) \equiv 0 \quad (i = 1, \ldots, n)$$

that is, the system (1.1) admits the autonomous motion

$$x_1 = 0, \qquad x_2 = 0, \ldots, x_n = 0 \tag{1.2}$$

The set of the n numbers $(x_1, \ldots, x_n)$ is called point x in the n-dimensional Euclidian space E^n. The number

$$\rho\,(x, x^\circ) = \sqrt{(x_1 - x_1^\circ)^2 + \ldots + (x_n - x_n^\circ)^2}$$
$$(\rho\,(x, M) = \inf\,[\rho\,(x, x^\circ),\ x^\circ \in M])$$

is called distance of the point x to the point x° in E^n (corresponding to the elements $M \subset E^n$).

The norm of the vector x appears as $\|x\| = \sqrt{(x_1^2 + \dots + x_n^2)}$. The set of functions

$$x(t, x_0, t_0) = \{x_1(t, x_{10}, \dots, x_{n0}, t_0), \dots, x_n(t, x_{10}, \dots, x_{n0}, t_0)\}$$

is determined in the domain (H)

$$\|x\| < H$$

of the space E^n of the nonautonomous motion.

We shall study the Liapunov functions $V(x, t)$, $W(x, t)$, determined (existing and well defined) and continuous in Γ, as well as their derivatives $\dot{V}(x, t)$, $\dot{W}(x, t)$, with respect to time t, taken by virtue of system (1.1), whereupon $V(0, t) \equiv 0$, $\dot{V}(0, t) \equiv 0$ (refer to [1]), and also the functions $V^*(x)$, $V'(x)$ are definite and continuous in (H).

The set of the points $x \in (H)$, for which $V^*(x) = 0$, will be represented by $E(V^* = 0)$.

Definition 1.1. $\dot{W}(x, t)$ is definitely not equal to zero in the ensemble $E(V^* = 0)$ if for any numbers α and $A (0 < \alpha < A < H)$ numbers $r_1(\alpha, A)$, $\xi(\alpha, A)$ $(0 < r_1 < d_1, \xi > 0)$, can be found such that $|\dot{W}(x, t)| > \xi$ for $\alpha < \|x\| < A$, $\rho(x, E(V^* = 0)) < r_1$, $t \geqslant 0$.

Theorem 1.1. Let there be functions $V(x, t)$, $W(x, t)$, having in Γ the following properties.

1) The function $V(x, t)$ is positive definite and admits an infinitely small upper limit.

2) The derivative $\dot{V}(x, t) \leqslant V^*(x) \leqslant 0$.

3) The function $W(x, t)$ is bounded.

4) $\dot{W}(x, t)$ is definitely not equal to zero in the ensemble $E(V^* = 0)$.

Then the autonomous motion (1.2) of the system (1.1) is asymptotically stable with respect to x_0, t_0.

Proof. Let us assume the conditions of the theorem are satisfied. By virtue of the theorem of Liapunov on the stability of motion, defined more precisely by Persidskii for the case of uniform stability, the autonomous motion (1.2) of the system (1.1) is stable with respect to t_0 and for any number $A (0 < A < H)$, a number $\lambda(A) (0 < \lambda < A)$ can be found such that for any

$$t_0 \geqslant 0, \quad \|x_0\| \leqslant \lambda \tag{1.3}$$

for all $t > t_0$ we shall have $\| x(t, x_0, t_0) \| < A$, whereupon

$$V(x_0, t_0) \leqslant \sup [V(x_0, t_0), t_0 \geqslant 0, \|x_0\| \leqslant \lambda] < \inf [V(x, t), t \geqslant 0, \|x\| = A] = V_A$$

There remains to prove that, to any arbitrary small number $\mu (0 < \mu < \lambda)$ there corresponds a positive number $T(A, \mu)$ such that under conditions (1.3) for all $t \geqslant t_0 + T$ there will be $\| x(t, x_0, t_0) \| < \mu$. Taking into consideration the monotonicity of the function $V(x(t, x_0, t_0), t)$, it is enough to establish the existence of a number $T^*(A, \mu)$ such that for any nonautonomous motion with initial condition (1.3) at the instant $t^* = t_0 + T^*$ there will be

$$V(x(t^*, x_0, t_0), t^*) < \inf [V(x, t), t \geqslant 0, \|x\| \geqslant \mu, \|x\| < A] = V_\mu$$

As $V(x, t)$ admits an infinitely small upper limit, then on V_μ a number $\alpha(\mu, A) (0 < \alpha < \mu)$ can be found such that for $t \geqslant 0$, $\|x\| \leqslant \alpha$ there will be $V(x, t) < V_\mu$. For this reason, to establish the existence of T^* it is enough to find a positive number $T'(\alpha, A)$ such that for any nonautonomous motion with initial conditions (1.3), at some instant of time $t' (t_0 \leqslant t' \leqslant t_0 + T')$, we shall have in it $\| x(t', x_0, t_0) \| \leqslant \alpha$.

We shall consider any nonautonomous motion $x(t) = x(t, x_0, t_0)$ with the initial condition (1.3) and we shall establish some of its properties

(a) If $\rho(x(t), x(\tau)) \geqslant r > 0 (t > \tau)$, then

$$t - \tau \geqslant \frac{r}{X \sqrt{n}}$$

From the formulas of finite increments

$$| x_i(t) - x_i(\tau) | = \left| \frac{dx_i}{dt} \right| (t - \tau) \leqslant X (t - \tau)$$

therefore, for $r \leqslant \sqrt{[x_1(t) - x_1(\tau)]^2 + \ldots + [x_n(t) - x_n(\tau)]^2}$ we have

$$r \leqslant X \sqrt{n} (t - \tau)$$

By virtue of (3) there exists a positive number

$$L = \sup (| W(x, t) |, t \geqslant 0, \|x\| < A)$$

In accordance with (4) it is possible to find some positive numbers $r_1(\alpha, A)$, $\xi(\alpha, A)$ such that in the ensemble $U \subset (H)$, where $\alpha < \|x\| < A$, $\rho(x, E(V^* = 0)) < r_1$, for any $t \geqslant 0$ there will be

$$| \dot{W}(x, t) | > \xi.$$

(b) The nonautonomous motion $x(t)$ cannot stay permanently in the set U, during an interval of time equal to $2L/\xi$.

Let us assume $x(\tau) \in U$. For $x(t)\,(t > \tau)$

$$W(t) - W(\tau) = \int_\tau^t \dot{W}\,dt$$

and as long as the motion of $x(t)$ is in U, the sign of the derivative $\dot{W}(t)$ does not change; therefore

$$|W(t)| + |W(\tau)| \geqslant \int_\tau^t |\dot{W}|\,dt > \xi(t-\tau)$$

But this inequality can be realized simultaneously with $|W| \leqslant L$ only for

$$t < \tau + 2L/\xi$$

Thus, there exists a number $\tau^*\,(\tau < \tau^* \leqslant \tau + 2L/\xi)$ such that for $t = \tau^*$ the motion is on the limit of the contour $[U]$ of the set U.

(c) If at the instant τ

$$\alpha < \|x(\tau)\| < A, \quad \rho(x(\tau), \quad E(V^* = 0)) < r_1/2$$

and for all $\tau \leqslant t \leqslant \tau + 2L/\xi$, there will be $\|x(t)\| > \alpha$, then at the instant $\tau^*\,(\tau < \tau^* \leqslant \tau + 2L/\xi)$, when

$$\alpha < \|x(\tau^*)\| < A, \qquad \rho(x(\tau^*), \qquad E(V^* = 0)) = r_1$$

we shall have

$$V(\tau^*) \leqslant V(\tau) - a(\alpha, A) \qquad a = \frac{\varepsilon r_1}{2X\sqrt{n}} > 0$$
$$\varepsilon = \inf\left[\,|V^*(x)|, \alpha < \|x\| < A, \rho(x, E(V^* = 0)) \geqslant r_1/2\right] > 0$$

Actually, under the given initial conditions, an instant $\tau_*\,(\tau < \tau_* < \tau^*)$ can be found such that

$$\alpha < \|x(\tau_*)\| < A, \qquad \rho(x(\tau_*), \ E(V^* = 0)) = r_1/2$$

and for all $\tau_* \leqslant t \leqslant \tau^*$, we shall have

$$\alpha < \|x(t)\| < A, \qquad r_1/2 \leqslant \rho(x(t), \qquad E(V^* = 0)) \leqslant r_1$$

There follows, in agreement with (2)

$$\dot{V}(t) \leqslant V^*(x(t)) \leqslant -\varepsilon$$

But, as is easily noticeable, $\rho(x(\tau^*), x(\tau_*)) \geqslant r_1/2$, whereupon, taking (a) and (b) into consideration

$$\frac{2L}{\xi} \geqslant \tau^* - \tau_* \geqslant \frac{r_1}{2X\sqrt{n}}$$

Therefore

$$V(\tau^*) - V(\tau) = \int_{\tau}^{\tau_*} \dot{V}\,dt + \int_{\tau_*}^{\tau^*} \dot{V}\,dt \leqslant \int_{\tau_*}^{\tau^*} \dot{V}(t)\,dt \leqslant -\varepsilon(\tau^* - \tau_*) \leqslant -\frac{\varepsilon r_1}{2X\sqrt{n}}$$

Let us consider the sequence of instants of time

$$t_k = t_0 + k2L/\xi \qquad (k = 0, 1, 2, \ldots)$$

(d) If the motion $x(t)$ on the interval of time $t_k \leqslant t \leqslant t_{k+2}$ is permanently in the domain $\alpha < \|x(t)\| < A$, then

$$V(t_{k+2}) \leqslant V(t_k) - a$$

In fact, if for $t_k \leqslant t \leqslant t_{k+1}$ we have constantly

$$\alpha < \|x(t)\| < A, \qquad \rho(x(t),\ E(V^* = 0)) \geqslant r_1/2$$

then

$$V(t_{k+2}) - V(t_k) \leqslant \int_{t_k}^{t_{k+1}} \dot{V}(t)\,dt \leqslant -\frac{2L\varepsilon}{\xi} \leqslant -a$$

But if we have $t_k \leqslant \tau \leqslant t_{k+1}$ such that

$$\alpha < \|x(\tau)\| < A, \quad \rho(x(\tau),\ E(V^* = 0)) < r_1/2$$

then in accordance with (b) a value $\tau^*(\tau < \tau^* \leqslant t_{k+2})$, can be found, in the presence of which

$$\rho(x(\tau^*)\ E(V^*(x) = 0)) = r_1$$

and in agreement with (c)

$$V(\tau^*) \leqslant V(\tau) - a \leqslant V(t_k) - a$$

and therefore also

$$V(t_{k+2}) \leqslant V(\tau^*) \leqslant V(t_k) - a$$

We shall choose an arbitrary integer $k' \geqslant V_A/a > 0$ and shall take $T' = t_{2k'}(\alpha, A)$. If it is assumed that for all $t_0 \leqslant t \leqslant t_0 + T'$ we have $\alpha < \|x(t)\| < A$, then in agreement with (d)

$$V(t_{2k'}) \leqslant V(x_0, t_0) - k'a < V_A \dot{-} k'a \leqslant 0$$

which is incompatible with the condition (1).

Thus, $t'(t_0 \leqslant t' \leqslant t_0 + T')$ can be found such that $\| x(t') \| \leqslant \alpha$, which proves the theorem.

We shall denote by $E_t(\dot{V} = 0)$ the set of points $x \in (H)$ for which $V(x, t) = 0$ for a given $t \in [0, \infty)$.

Definition 1.2. $\dot{W}(x, t)$ is definitely not equal to zero in the sets $E_t(\dot{V} = 0)$ if for any numbers α and $A(0 < \alpha < A < H)$ positive numbers $l(\alpha, A)$, $\xi(\alpha, A)$ can be found such that

$$|\dot{W}(x, t)| > \xi \text{ for } \alpha < \|x\| < A,\ |\dot{V}(x\ t)| < l,\ t \geqslant$$

Theorem 1.2. Let there be functions $V(x, t)$, $W(x, t)$ having in Γ the following properties.

1) The function $V(x, t)$ is positive definite.

2) The derivative $\dot{V}(x, t) \leqslant 0$, and the partial derivatives $\partial V/\partial x_s$, $\partial V/\partial t$, $\partial^2 V/\partial x_s \partial x_i$, $\partial^2 V/\partial x_s \partial t$, $\partial^2 V/\partial t^2$ are continuous and bounded.

3) The function $W(x, t)$ is bounded.

4) $\dot{W}(x, t)$ is definitely not equal to zero in the sets $E_t(\dot{V} = 0)$.

Then the autonomous motion (1.2) of the system (1.1) is asymptotically stable with respect to x_0, t_0.

Note. Theorem (1.1) for $X_i(x, t)$ continuous in (H) with respect to $t \in [0, \infty)$ and Theorem (1.2) admit some simplifications.

In fact, if the autonomous motion (1.2) of the system (1.1) is asymptotically stable with respect to x_0, t_0, then, as shown by Malkin [9], in some neighborhood of the autonomous motion $(H_0)[(H_0)] \subset (H)$ $t \in (0, \infty)$, there exists a positive definite function $V(x, t)$ admitting an infinitely small higher limit, and the derivative of which, in accordance with (1.1), is negative definite. If, furthermore, the functions $X_i(x, t)$ are continuous in (H) with respect to $t \in (0, \infty)$, then, as shown by Krasovskii [4], the mentioned function $V(x, t)$ has continuous partial derivatives of any order with respect to all variables, whereupon these derivatives are uniformly bounded in the domain (H_0) for $t \in (0, \infty)$. Taking, for instance, $W(x, t) = V(x, t)$, we get two functions, satisfying for $x \in (H_0)$, $t \in (0, \infty)$, the conditions of Theorems (1.1) and (1.2). As we consider the asymptotic stability in the sense of Liapunov [1, 9, 10] (local), we get a contraction of the domain of existence of the function, which is not important compared with Γ.

Definition 1.3. $\dot{W}(x, t)$ is strictly not equal to zero in the set $E(V^* = 0)$, if for any numbers α and $A(0 < \alpha < A < H)$ it is possible to find a number $r_1(\alpha, A)(0 < r_1 < \alpha)$ and a continuous function $\xi_\alpha(t)$ such that for any $t \geqslant 0$

$$\xi_\alpha(t) > 0, \int_t^\infty \xi_\alpha(\tau)\, d\tau = \infty \tag{1.4}$$

and in the set, where $\alpha < \|x\| < A$, $\rho(x, E(V^* = 0)) < r_1$, $t \geqslant 0$, we shall have

$$|\dot{W}(x, t)| \geqslant \xi_\alpha(t)$$

Theorem 1.3. Let there be functions $V(x, t)$, $W(x, t)$ having in Γ the following properties:

1) The function $V(x, t)$ is positive definite and admits an infinitely small higher limit.

2) The derivative $\dot{V}(x, t) \leqslant 0$ and in each domain $t \geqslant 0$, $\alpha < \|x\| < H$, there will be $\dot{V}(x, t) \leqslant \varphi_\alpha(t) V^*(x)$, where $V^*(x) \leqslant 0$ and $\varphi_\alpha(t)$ is a continuous non-negative function of t such that for any infinite system S of closed non-intersecting intervals of the semi-axis $[0, \infty)$ of an identical fixed positive length, we have

$$\int_S \varphi_\alpha(t)\, dt = \infty$$

3) The function $W(x, t)$ is bounded.

4) $\dot{W}(x, t)$ is strictly not equal to zero in the set $E(V^* = 0)$.

Then the autonomous motion (1.2) of the system (1.1) is asymptotically stable with respect to x_0.

Definition 1.4. $\dot{W}(x, t)$ is strictly not equal to zero in the sets $E_t(\dot{V}, = 0)$, if for any numbers α and $A(0 < \alpha < A < H)$ a positive number $l(\alpha, A)$ and a continuous function $\xi_\alpha(t)$ can be found such that for any $t \geqslant 0$

$$\xi_\alpha(t) > 0, \quad \int_t^\infty \xi_\alpha(\tau)\, d\tau = \infty$$

and in the set, where $\alpha < \|x\| < A$, $|V(x, t)| < l$, $t \geqslant 0$, we shall have

$$|\dot{W}(x, t)| \geqslant \xi_\alpha(t)$$

Theorem 1.4. Let there be functions $V(x, t)$, $W(x, t)$ having in Γ the following properties:

1) The function $\dot{V}(x, t)$ is positive definite.

2) The derivative $\dot{V}(x,\ t) \leqslant 0$, and the partial derivatives $\partial V/\partial x_s$, $\partial V/\partial t$, $\partial^2 V/\partial x_s \partial x_i$, $\partial^2 V/\partial x_s \partial t$, $\partial^2 V/\partial t^2$ are continuous and bounded.

3) The function $W(x,\ t)$ is bounded.

4) $\dot{W}(x,\ t)$ is strictly not equal to zero in the sets $E_t(\dot{V} = 0)$.

Then the autonomous motion (1.2) of the system (1.1) is asymptotically stable with respect to x_0.

Definition 1.5. The function $W(x,\ t)$ admits a higher limit, infinitely small in the set $E(V^* = 0)$, if it is limited, $W(x,\ t) = 0$ for $x \in E(V^* = 0)$, $t \geqslant 0$, and if for any given small numbers l, α, $A (l > 0,$ $0 < \alpha < A < H)$ a positive number r' can be found, such that for $\alpha < \|x\| < A$, $\rho(x,\ E(V^* = 0)) < r'$, $t \geqslant 0$ we shall have $|W(x,\ t)| < l$.

Theorem 1.5. Let there exist functions $V(x,\ t)$, $W(x,\ t)$, having in Γ the following properties:

1) The function $V(x,\ t)$ is positive definite.

2) The derivative $\dot{V}(x,\ t) \leqslant 0$ and in each domain $t \geqslant 0$, $\alpha < \|x\| < H$ we shall have $\dot{V}(x,\ t) \leqslant \varphi_\alpha(t) V^*(x)$, where $V^*(x) \leqslant 0$, and $\varphi_\alpha(t)$ is a continuous non-negative function such that for any given infinite system S of closed non-intersecting intervals of the semi-axis $[0,\ \infty)$ of an identical fixed positive length, we have

$$\int_S \varphi_\alpha(t)\, dt = \infty$$

3) The function $W(x,\ t)$ admits a higher limit, infinitely small in the set $E(V^* = 0)$.

4) $\dot{W}(x,\ t)$ is definitely not equal to zero in the set $E(V^* = 0)$.

Then the autonomous motion (1.2) of the system (1.1) is asymptotically stable.

The proofs of Theorems (1.2) to (1.5) come as modifications of the proof of Theorem (1.1).

The preceding definitions can be generalized to the case of "global" stability, if analogously to [4], an estimate of the domain of attraction of the autonomous motion is introduced in the conditions of the theorems.

2. *Theorem 2.1.* Let there be functions $V(x,\ t)$, $W(x,\ t)$ possessing in Γ the following properties:

1) The function $V(x,\ t)$ admits an infinitely small higher limit and

for any $t \geqslant 0$ it is possible to find points x lying in any given small neighborhood of the autonomous motion and such that in them $V(x, t) > 0$.

2) The derivative $\dot{V}(x, t) \geqslant 0$ and in each domain $t \geqslant 0$, $\alpha < \|x\| < A < H$ there will be $\dot{V}(x, t) \geqslant \varphi_\alpha(t)V'(x)$, where $V'(x) \geqslant 0$ and $\varphi_\alpha(t)$ is a continuous non-negative function of t such that for any infinite system S of closed non-intersecting intervals of the semi-axis $[0, \infty)$ of an identical fixed interval, we have

$$\int_S \varphi_\alpha(t)\, dt = \infty$$

3) The function $W(x, t)$ is bounded.

4) $\dot{W}(x, t)$ is strictly not equal to zero in the set $E(V' = 0)$.

Then the autonomous motion (1.2) of the system (1.1) is unstable.

Proof. We shall assume that the conditions of the theorem are satisfied, but the autonomous motion is table, i.e. for an A and t_0 a $\lambda > 0$ can be found such that in each nonautonomous motion, with the initial values (1.3) for $t \geqslant t_0$, we shall have

$$\|x(t, x_0, t_0)\| < A < H$$

Under the conditions (1.3) in accordance with (1), x_0^*, t_0 can be found such that

$$V(x_0^*, t_0) > 0$$

We shall consider the disturbed motion $x(t) = x(t, x_0^*, t_0)$ and its properties.

(a) If $\rho(x(t), x(\tau)) \geqslant r$ for $t > \tau$, then $t - \tau \geqslant r/X\sqrt{n}$.

(b) For every $t > t_0$ there will be $\alpha < \|x(t)\| < A$, where α is some positive number.

Actually, this is in agreement with our assumption $\|x(t)\| < A$, but in such a case $\dot{V} \geqslant 0$, i.e. $V(t) \geqslant V(x_0^*, t_0) > 0$. As $V(x, t)$ admits an infinitely small higher limit, then for the numbers $V(x_0^*, t_0) > 0$ a number $\alpha > 0$ will be found such that for all $t \geqslant t_0$, $\|x\| \leqslant \alpha$ we shall have $V(x, t) < V(x_0^*, t_0)$; consequently, $\|x(t)\| \leqslant \alpha$ is not possible.

In accordance with (4), a number $r_1(\alpha, A)(0 < r_1 < \alpha)$ and a continuous function $\xi_\alpha(t)$, satisfying the conditions (1.4) can be found such that for $\rho(x(t), E(V' = 0)) < r_1$, we shall have

$$|\dot{W}(t)| \geqslant \xi_\alpha(t).$$

(c) If $\rho(x(\tau), E(V' = 0)) < r_1$, then $\tau^* > \tau$ will be found such that

$$\rho\,(x\,(\tau^*),\ E\,(V'=0))=r_1$$

For $\rho(x(\tau),\ E(V' = 0)) < r_1$ for $x(t)\ \ (t > \tau)$

$$W\,(t)-W\,(\tau)=\int_{\tau}^{t}\dot{W}\,dt$$

and while $\rho(x(t),\ E(V' = 0)) < r_1$, the derivative $\dot{W}(t)$ does not change its sign; therefore,

$$|\,W\,(t)\,|+|\,W\,(\tau)\,|\geqslant\int_{\tau}^{t}|\,\dot{W}\,|\,dt\geqslant\int_{\tau}^{t}\xi_{\alpha}\,(t)\,dt$$

But, by virtue of (1.4) and the boundedness of $W(x,\ t)$, this cannot be for all $t > t_0$.

(d) If $\rho(x(\tau),\ E(V' = 0)) < r_1/2$, then for $t = \tau^*$, when $\rho(x(\tau^*),\ E(V' = 0)) = r_1$

$$V\,(\tau^*)\geqslant V\,(\tau)+\varepsilon'\int_{\tau^{**}}^{\tau^*}\varphi_{\alpha}\,(t)\,dt \qquad \left(\tau\leqslant\tau^{**}=\tau^*-\frac{r_1}{2X\sqrt{n}}\right)$$

$$\varepsilon'=\inf\,[V'\,(x),\ \alpha<\|x\|<A,\ \rho\,(x,\,E\,(V'=0))\geqslant r_1/2]>0$$

In fact, under the given conditions, $\tau < \tau_* < \tau^*$ can be found such that

$$\rho\,(x\,(\tau_*),\ E\,(V'=0))=r_1/2$$

and for $\tau_* \leqslant t \leqslant \tau^*$ we shall have

$$r_1/2\leqslant\rho\,(x\,(t),\,E\,(V'=0))\leqslant r_1$$

i.e. in accordance with (2)

$$\dot{V}\,(t)\geqslant\varphi_{\alpha}\,(t)\,V'\,(x\,(t))\geqslant\varepsilon'\varphi_{\alpha}\,(t)$$

Therefore

$$V\,(\tau^*)-V\,(\tau)\geqslant\varepsilon'\int_{\tau_*}^{\tau^*}\varphi_{\alpha}\,(t)\,dt$$

But, as it is easy to notice, $\rho(x(\tau^*),\ x(\tau_*)) \geqslant r_1/2$, whereupon, in agreement with (a),

$$\tau^*-\tau\geqslant\tau^*-\tau_*\geqslant\frac{r_1}{2X\sqrt{n}}$$

(e) There is not any number $\tau^0 \geqslant t_0$ such that for all $t > \tau^0$ we would have

$$\rho(x(t), E(V' = 0)) \geqslant r_1/2$$

In fact, if such a τ° existed, then for all $t > \tau^{\circ}$ we would have

$$V(t) = V(\tau^{\circ}) + \int_{\tau^{\circ}}^{t} \dot{V}\, dt \geqslant V(\tau^{\circ}) + \varepsilon' \int_{\tau^{\circ}}^{t} \varphi_\alpha(t)\, dt$$

and in accordance with (2) we have $V(t) \to \infty$ for $t \to \infty$, which is not compatible with the condition of the boundedness of $V(t)$, resulting from (1) and (b).

In accordance with (e) for any τ_i^*, a $\tau_{i+1} > \tau_i^*$ can be found such that

$$\rho(x(\tau_{i+1}), E(V' = 0)) < r_1/2$$

In accordance with (c) there corresponds to it $\tau_{i+1}^* > \tau_{i+1}$ such that

$$\rho(x(\tau_{i+1}^*), E(V' = 0)) = r_1$$

We shall consider the infinite sequence of numbers

$$t_0 < \tau_1 < \tau_1^* < \ldots < \tau_i < \tau_i^* < \ldots$$

In accordance with (2) and (d)

$$V(\tau_i^*) \geqslant V(t_0) + \varepsilon' \sum_{j=1}^{i} \int_{\tau_j^{**}}^{\tau_j^*} \varphi_\alpha(t)\, dt \qquad \left(\tau_j \leqslant \tau_j^{**} = \tau_j^* - \frac{r_1}{2X\sqrt{n}}\right)$$

The infinite system of segments $[\tau_i^{**}, \tau_i^*]$ satisfies the condition of (2) for the system S, therefore, the last sum increases indefinitely with i, i.e. $V(\tau_i^*) \to \infty$ for $i \to \infty$. But this is incompatible with the condition of boundedness of the function V. The contradiction shows that the assumption of stability is wrong, which proves the theorem.

Theorem 2.2. Let there be functions $V(x, t)$, $W(x, t)$ having in Γ the following properties:

1) The function $V(x, t)$ is such that for any $t \geqslant 0$ it is possible to find points x, lying in any small neighborhood of the autonomous motion, in which $V(x, t) > 0$.

2) The derivative $\dot{V}(x, t) \geqslant 0$ and the partial derivatives $\partial V/\partial x_i$, $\partial V/\partial t$, $\partial^2 V/\partial x_i \partial x_j$, $\partial^2 V/\partial x_i \partial t$, $\partial^2 V/\partial t^2$ are continuous and bounded.

3) The function $W(x, t)$ is bounded.

4) $\dot{W}(x, t)$ is strictly not equal to zero in the sets $E_t(\dot{V} = 0)$.

Then the autonomous motion (1.2) of the systems (1.1) is unstable.

The proof comes as a modification of the previous one.

The autonomous motion (1.2) of the system (1.1) is called absolutely unstable if, for any A, x_0, t_0 satisfying the conditions $0 < A < H$, $0 < \| x_0 \| < A$, $t_0 \geqslant 0$, a positive number T can be found such that $\| x(t_0 + T, x_0, t_0) \| = A$. It is easy to prove the following generalization of the criterion of Dubovshin [10]. If the conditions of Theorems (2.1) or (2.2) are satisfied and $V(x, t) > 0$ for all $t \geqslant 0$, $0 < \|x\| < H$, then the autonomous motion (1.2) of the system (1.1) is absolutely unstable.

3. Let us consider some applications. A. A symmetrical, heavy, rigid body with one point fixed, under the presence of resistance forces of the medium.

With the usual designation, the equations of motion have the form

$$\begin{aligned} A\dot{p} + (C - A)\,qr &= Pz_0\gamma_2 - \partial R / \partial p, & \dot{\gamma}_1 &= r\gamma_2 - q\gamma_3 \\ A\dot{q} + (A - C)\,pr &= -Pz_0\gamma_1 - \partial R / \partial q, & \dot{\gamma}_2 &= p\gamma_3 - r\gamma_1 \\ C\dot{r} &= M_z(t, r), & \dot{\gamma}_3 &= q\gamma_1 - p\gamma_2 \end{aligned}$$

Here R is a homogeneous function of p, q of order $m \geqslant 2$, the coefficients of which are continuous and limited functions of t. Let the moment M_z with respect to the axis of symmetry z consist of the moments of the resistance forces, depending on r, t and the moment of the driving force given as a function of t.

The third equation determines r as a function of time $r(t, r_0, t_0)$ which we shall assume continuous and bounded. The equations of motion admit the solution

$$p = 0,\ q = 0,\ r = r(t, r_0, t_0),\ \gamma_1 = 0,\ \gamma_2 = 0,\ \gamma_3 = 1 \tag{3.1}$$

describing the irregular rotation of the body around the vertical axis of symmetry.

The functions

$$V = \frac{1}{2} A (p^2 + q^2) - \frac{1}{2} Pz_0 (\gamma_1^2 + \gamma_2^2 + \gamma^4),\ W = A (p\gamma_2 - q\gamma_1) \qquad (\gamma^2 = 1 - \gamma_3 \geqslant 0)$$

have, by virtue of the equations of the nonautonomous motion (taking into consideration the trigonometric relation $\gamma_1^2 + \gamma_2^2 + \gamma_3^2 = 1$), derivatives with respect to time

$$\dot{V} = -mR \leqslant 0,$$

$$\dot{W} = \frac{1}{2} Pz_0 [\gamma_1^2 + \gamma_2^2 + \gamma^2 (2 - \gamma^2)] - \gamma_1 \left(Cpr - \frac{\partial R}{\partial q}\right) - \gamma_2 \left(Cqr + \frac{\partial R}{\partial p}\right) + A\gamma_3 (p^2 + q^2)$$

The set $E(\dot{V} = 0)$ corresponds to $p = 0$, $q = 0$. In it

$$\dot{W} = \frac{1}{2} Pz_0 [\gamma_1^2 + \gamma_2^2 + \gamma^2 (2 - \gamma^2)]$$

If $z_0 \neq 0$, then

$$|\dot{W}| > \frac{1}{2} P |z_0| \alpha^2 \text{ for } p = 0,\ q = 0,\ 0 < \alpha^2 < \gamma_1^2 + \gamma_2^2 + \gamma^2 < A^2 < H^2$$

Therefore, by virtue of the continuity of $\dot{W}$ and the boundaries of the coefficients, it is possible to find $r_1 > 0$, such that

$$|\dot{W}| > \frac{1}{4} P |z_0| \alpha^2 \text{when } t \geqslant 0,\ \alpha^2 < \gamma_1^2 + \gamma_2^2 + \gamma^2 < A^2,\ p^2 + q^2 < r_1^2$$

i.e. $\dot{W}$ is definitely not equal to zero in the set $E(\dot{V} = 0)$.

If $z_0 < 0$ (the center of gravity is lower than the point of support), then the conditions of Theorem (1.1), on the basis of which we conclude the asymptotic stability (with respect to p_0, q_0, γ_{10}, γ_{20}, γ_0, t_0) of the autonomous motion (3.1), are satisfied.

If on the contrary $z_0 > 0$, then the conditions of Theorem (2.1) are satisfied and the autonomous motions is unstable.

B. A nonstationary mechanical system under the action of potential, gyroscopic and dissipative forces

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{n} g_{ij} \dot{q}_j + \frac{\partial U}{\partial q_i} - \frac{\partial R}{\partial \dot{q}_i} \quad (i = 1, \ldots, n) \tag{3.2}$$

Here $q = (q_1, \ldots, q_n)$, $\dot{q} = (\dot{q}_1, \ldots, \dot{q}_n)$ are combinations of the generalized coordinates and velocities; T is a positive definite quadratic form of the velocities; $g_{ij}(q, t) = -g_{ij}(q, t)$ are the gyroscopic coefficients; R is the function of dissipation and a positive definite quadratic form of $\dot{q}$ (full dissipation); $U(q)$ is the forcing function, which we shall assume to be a holomorphic function of q

$$T = \frac{1}{2} \sum_{i,\, j=1}^{n} a_{ij} \dot{q}_i \dot{q}_j, \qquad R = \frac{1}{2} \sum_{i,\, j=1}^{n} b_{ij} \dot{q}_i \dot{q}_j \qquad \begin{pmatrix} a_{ij}(q) = a_{ji}(q) \\ b_{ij}(q, t) = b_{ji}(q, t) \end{pmatrix}$$

$$R \geqslant \frac{1}{2} \beta \sum_{i=1}^{n} \dot{q}_i^2 \qquad (\beta > 0),\ U(q) = \sum_{k=m}^{\infty} U_k \ (m \geqslant 2)$$

where $U_k(q)$ is a homogeneous function of order k.

Further, g_{ij}, b_{ij} are assumed to be holomorphic functions of q with continuous and bounded coefficients.

The system (3.2) admits the solution

$$q_1 = 0, \ldots, q_n = 0, \ \dot{q}_1 = 0, \ldots, \dot{q}_n = 0 \tag{3.3}$$

Taking $V = T - U$, we have, by virtue of (3.2)

$$\dot{V} = -2R \leqslant -\beta(\dot{q}_1^2 + \ldots + \dot{q}_n^2) \leqslant 0$$

We shall take

$$W = \sum_{i=1}^{n} \frac{\partial T}{\partial \dot{q}_i} q_i$$

By virtue of (3.2)

$$\dot{W} = 2T + \sum_{i,\,j=1}^{n} g_{ij} q_i \dot{q}_j + \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} q_i + \sum_{i=1}^{n} q_i \left(\frac{\partial T}{\partial q_i} - \frac{\partial R}{\partial \dot{q}_i} \right)$$

The set $E(\beta(\dot{q}_1^2, + \ldots + \dot{q}_n^2) = 0)$ is determined as

$$\dot{q}_1 = 0, \ldots, \dot{q}_n = 0, \ q_1^2 + \ldots + q_n^2 < H^2$$

In it

$$\dot{W} = \sum_{i=1}^{n} \frac{\partial U}{\partial q_i} q_i = \sum_{k=m}^{\infty} kU_k$$

If this function is sign-definite, then for any α and $A(0 < \alpha < A < H)$, a number $\xi > 0$ can be found such that

$$|\dot{W}| > 2\xi > 0 \text{ when } \dot{q}_1 = 0, \ \dot{q}_2 = 0, \ldots, \dot{q}_n = 0, \ \alpha^2 < \sum_{i=1}^{n} q_i^2 < A^2$$

But by virtue of the continuity of $\dot{W}$ and the boundedness of the coefficients, it is possible to find for ξ, an $r_1 > 0$ such that we shall have $|\dot{W}| > \xi > 0$ for

$$\alpha^2 < \sum_{i=1}^{n} (q_i^2 + \dot{q}_i^2) < A^2, \ \sum_{i=1}^{n} \dot{q}_i^2 < r_1^2, \ t \geqslant 0$$

i.e. $\dot{W}$ is definitely not equal to zero in the set $E(\beta(\dot{q}_1^2 + \cdots + \dot{q}_n^2$ $(= 0))$.

If the functions

$$U(q),\quad \sum_{k=m}^{\infty} kU_k(q)$$

are negative definite, then the conditions of Theorem (1.1), on the basis of which the autonomous motion (3.3) of the asymptotic stability is uniform with respect to the variables q_0, $\dot{q}_0$, t_p, are satisfied.

If on the contrary $U(q)$ can take positive values for any arbitrarily small $|q_1|, \ldots, |q_n|$, and the function

$$\sum_{k=m}^{\infty} kU_k(q)$$

is sign-definite, then the conditions of Theorem (2.1) are satisfied and by this theorem the autonomous motion is unstable.

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